THREE-BODY ENCOUNTER RATE IN ONE DIMENSION

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Abstract. A 3-body encounter is defined as the moment in time at which the largest pairwise separation among the three bodies is the smallest, and such an encounter is considered "interesting" if this separation is below some threshold ϵ. I calculate the rate of 3-body encounters undergone by a reference body with velocity v_0, given that all bodies move at constant speeds in one dimension, their density is n per unit length, and the probability density function of their velocities is $f(v)$. The result is that the rate linearly depends on ϵ and quadratically on n. The dependence of the 3-body encounter rate on the velocity v_0 is calculated analytically for a simple model representing pedestrian movement in a street. For this model, the encounter rate is the lowest at low velocities of the reference body, and increases linearly for high velocities.

1. Formulation of the problem

A one-dimensional space is occupied by bodies at a density of n per unit length. Each body has a constant speed drawn from a velocity distribution $f(v)$, where v could be positive or negative. Given that the velocity of one of the bodies (henceforth the reference body) is v_0, the goal is to compute the 3-body encounter rate (i.e. encounters per unit time), defined mathematically below. This could be applied, for example, to pedestrians walking in a street, where a 3-body encounter corresponds to one pedestrian passing another, when at almost the same time being passed by a third from behind, or encountering a third coming from the opposite direction. The simplifying assumptions are that each body moves independently at a constant speed, and encounters do not induce a change.

Let us denote the velocity of body i by v_i and its position at time $t = 0$ as a_i. The equation of motion is therefore $x_i(t) = a_i + v_i t$. The path is a linear function in x-t space. A 2-body encounter can be more simply defined as the time of crossing of two paths. In mathematical terms, an 2-body encounter between pedestrians i and j occurs at some time $t_{\rm enc}$ when $x_i(t_{\rm enc}) = x_j(t_{\rm enc})$ is satisfied. A 3-body encounter is a more complicated concept. Three lines generally do not cross at the same point (unless they are finely tuned to do so). However, for each triplet of lines we can still find a time in which the maximum separation between each pair is the smallest. In mathematical terms, a 3-body encounter of bodies i, j and k occurs at some time $t_{\rm enc}$ when the function

$$s(t) = \max(|x_i - x_j|, |x_i - x_k|, |x_j - x_k|)$$

has a minimum. Unlike a 2-body encounter, a 3-body encounter will have a property $\delta \equiv s(t_{\rm enc})$ which I call the *encounter parameter*. An encounter is considered "interesting" if δ is smaller than some threshold ϵ.

2. Encounter time and parameter

Let us consider three bodies, their initial positions and speeds are subscriptted with i, j, and k. We can assume without loss of generality that $v_i > v_j > v_k$. Among these three bodies, there are of

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course three 2-body encounters, the times are

$$t_{ij} = -\frac{a_i - a_j}{v_i - v_j}$$
$$t_{ik} = -\frac{a_i - a_k}{v_i - v_k} \tag{2.1}$$
$$t_{jk} = -\frac{a_j - a_k}{v_j - v_k}$$

We can show that t_{ik} always lies between the other two. We do this by writing $t_{ik} = \alpha t_{ij} + (1-\alpha)t_{jk}$ where $\alpha = (v_i - v_j)/(v_i - v_k) < 1$. We do not know which of t_{ij} or t_{jk} is larger; both cases are possible. Let us consider first the case where $t_{ij} < t_{ik} < t_{jk}$. The function s is

$$s(t) = \begin{cases} x_k - x_i = a_k - a_i + (v_k - v_i)t & t < t_{ij} \\ x_k - x_j = a_k - a_j + (v_k - v_j)t & t_{ij} < t < t_{ik} \\ x_i - x_j = a_i - a_j + (v_i - v_j)t & t_{ik} < t < t_{jk} \\ x_i - x_k = a_i - a_k + (v_i - v_k)t & t_{jk} < t \end{cases}$$

Each segment of $s(t)$ is a straight line; the first two have negative slopes and the last two have positive slopes. Therefore the minimum must be exactly at t_{ik}, which means the encounter time is $t_{\text{enc}} = t_{ik}$ and the encounter parameter $\delta = s(t_{ik})$ can be found by substituting t_{ik} into the second or third segments of $s(t)$. By doing so and also considering the opposite case where $t_{jk} < t_{ik} < t_{ij}$, we find that

$$\delta = \frac{1}{v_i - v_k} \left| v_i(a_j - a_k) + v_j(a_k - a_i) + v_k(a_i - a_j) \right| . \tag{2.2}$$

3. Encounter rate at fixed velocities

Given the speed v_0 of the reference body and the speeds v_1 and v_2 of two other bodies, what is the 3-body encounter rate? We need to consider three separate cases, namely that the reference body is i, j, or k. In other words, weather the reference body has the highest speed in the positive direction, or it is the second or third in that order. The method of finding the rate is based on counting the number of encounters with $\delta < \epsilon$ between some time t_0 and another time $t_0 + \Delta t$; the rate would be the number of encounters in this period divided by Δt. The reference body has some specific initial position a_0 (we do not expect the rate to depend on it), the initial positions of the two other bodies a_1 and a_2 can be anything. Some value combinations will satisfy the conditions $\delta < \epsilon$ and $t_0 < t_{\text{enc}} < t_0 + \Delta t$ while others will not. These two conditions define a region in a_1-a_2 space and its area is proportional to the number of encounters. If the area is A, the rate is

$$r = \frac{n^2 A}{\Delta t} \tag{3.1}$$

3.1. Reference body is *i*.

It is easy to see from Equations (2.1) and (2.2) that the region is a parallelogram in a_j-a_k space with one side parallel to the a_j-axis the other one has slope of $(v_i - v_k)/(v_i - v_j)$. To calculate the area, we just find the height and base of the parallelogram. The height is simply $\Delta a_k = \Delta t(v_i - v_k)$ and the base is $\Delta a_j = 2\epsilon$. The area is therefore

$$A_i = 2\epsilon \Delta t(v_i - v_k). \tag{3.2}$$

3.2. Reference body is *j*.

In this case, from the same equations we can see that the region is also a parallelogram in a_i-a_k space. One side is at 45° and the other has a slope of $-(v_i - v_k)/(v_i - v_j)$. This time we calculate the area by constructing a rectangle with edges parallel to the axes and that contains the parallelogram. The relevant area is then the area of this rectangle minus the areas of two pairs of identical triangles. In mathematical terms

$$A_j = (\Delta a_i + b_i)(\Delta a_k + b_k) - 2{\cdot}\tfrac{1}{2}\Delta a_i \Delta a_k - 2{\cdot}\tfrac{1}{2} b_i b_k = \Delta a_i b_k + \Delta a_k b_i$$

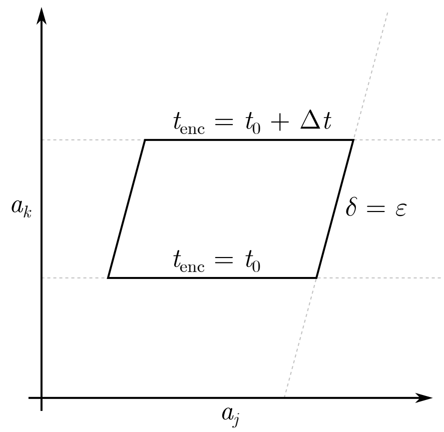

FIGURE 3.1. For the case where the reference body is i (largest speed in the positive direction), the area in a_j-a_k space where the encounters satisfy $\delta < \epsilon$ and $t_0 < t_{\rm enc} < t_0 + \Delta t$ is a parallelogram as shown.

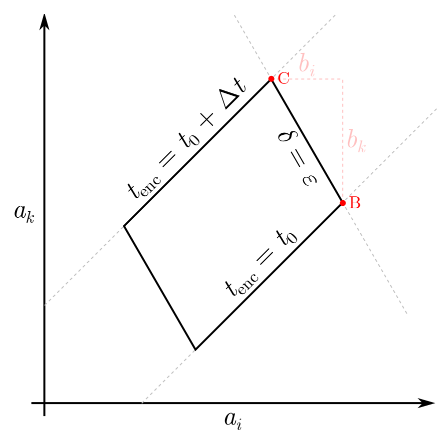

FIGURE 3.2. Same as Figure 3.1 for the case where the reference body is j.

where b_i and b_k are auxiliary quantities defined in Figure 3.2. The quantities:Δa_i and Δa_k are identical and equal 2ϵ because of the $45°$ slope. We find b_i and b_k from the coordinates of points B and C shown in the figure. B is the intersection of $t_{\rm enc} = t_0$ and $\delta = \epsilon$, which is at

$$a_{i,\rm B} = a_j - \epsilon - (v_i - v_j)t_0$$
$$a_{k,\rm B} = a_j - \epsilon + (v_j - v_k)t_0$$

The point C is the same with $t_0 \to t_0 + \Delta t$. Thus,

$$b_i = a_{i,\rm B} - a_{i,\rm C} = (v_i - v_j)\Delta t$$
$$b_k = a_{k,\rm C} - a_{k,\rm B} = (v_j - v_k)\Delta t$$

and finally

$$A_j = 2\epsilon\Delta t(v_i - v_k) \tag{3.3}$$

as in the previous case.

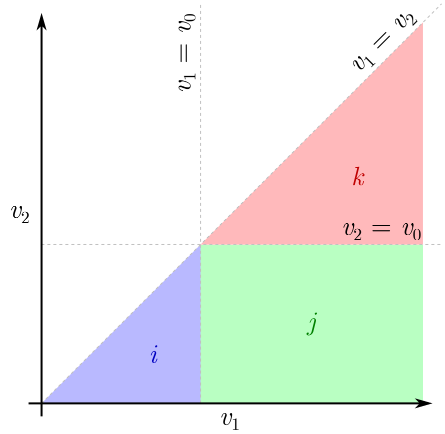

FIGURE 4.1. A diagram showing the different cases for $v_1 > v_2$: the shaded regions show where the reference body is i (blue), j (green), and k (red).

3.3. **Reference body is k.** This is very similar to the first case, and just from symmetry considerations we can see that the result here is the same

$$A_k = 2\epsilon\Delta t(v_i - v_k) \tag{3.4}$$

4. Integrating over velocities

For each of the three cases discussed above, there are two sub-cases, namely $v_1 > v_2$ and $v_2 > v_1$ (regardless of the position of v_0 in the order). Without loss of generality we assume the former; from symmetry the rate when the opposite is true is equal, so the result should be multiplied by two. From Equation (3.1) and the results for A given in Equations (3.2), (3.3), and (3.4) we know the rate given any pair (v_1, v_2). The joint probability to draw two velocities in an infinitesimal environment $\mathrm{d}v_1\mathrm{d}v_2$ around (v_1, v_2) is $f(v_1)f(v_2)\mathrm{d}v_1\mathrm{d}v_2$, which is the weight given to the rate calculated from these velocities. The integration limits are easy to see in Figure 4.1. The general expression for the 3-body encounter rate is thus

$$\begin{aligned} r_{3\mathrm{b}} = 4n^2\epsilon\Bigg[&\int_{-\infty}^{v_0} \mathrm{d}v_2 \int_{v_2}^{v_0} \mathrm{d}v_1\, f(v_1)f(v_2)(v_0 - v_2) \\ &+ \int_{-\infty}^{v_0} \mathrm{d}v_2 \int_{v_0}^{\infty} \mathrm{d}v_1\, f(v_1)f(v_2)(v_1 - v_2) \\ &+ \int_{v_0}^{\infty} \mathrm{d}v_2 \int_{v_0}^{v_2} \mathrm{d}v_1\, f(v_1)f(v_2)(v_1 - v_0)\Bigg]. \end{aligned}$$

We can somewhat simplify this as follows:

$$r_{3\mathrm{b}} = 4n^2\epsilon\left[\int_{-\infty}^{\infty} |v - v_0||F(v) - F(v_0)|f(v)\mathrm{d}v - \int_{-\infty}^{v_0} vf(v)\mathrm{d}v + F(v_0)\langle v\rangle\right] \tag{4.1}$$

where

$$F(v) \equiv \int_{-\infty}^{v} f(v')\mathrm{d}v'$$

is the cumulative distribution function and

$$\langle v\rangle \equiv \int_{-\infty}^{\infty} v' f(v')\mathrm{d}v'$$

is the average velocity.

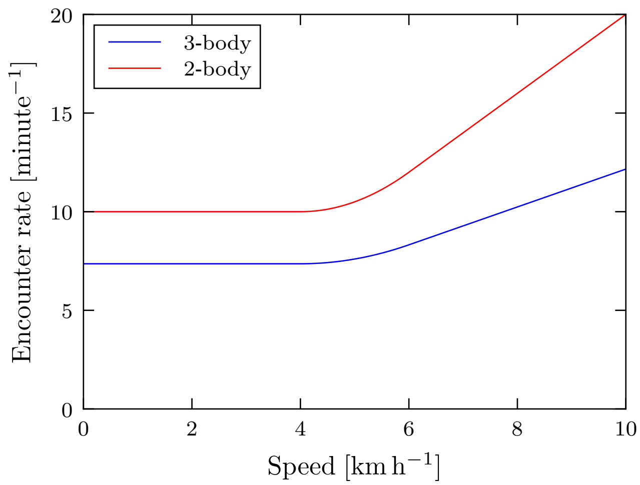

FIGURE 6.1. The 2- and 3-body encounter rate as a function of the reference body's speed, and where the other bodies' velocity distribution is a made up of two rectangular functions with widths w around $\pm V$. For the purpose of this plot we assumed $V = 5\,\mathrm{km\,h^{-1}}$, $w = 2\,\mathrm{km\,h^{-1}}$, $n = 120\,\mathrm{km^{-1}}$ (representing a mildly busy street in Budapest), and $\epsilon = 0.5\,\mathrm{m}$ (representing a lower limit on comfortable personal space).

5. Rate of 2-body encounters

For comparison it is useful to also consider the 2-body encounter rate. Let us first assume that all bodies have the same velocity v except the reference body, which has velocity v_0. The encounter time with a particular body with initial position a is

$$t_{\mathrm{enc}} = -\frac{a_0 - a}{v_0 - v}$$

As the average distance between pedestrians on the x-axis at $t = 0$ is $\Delta a = n^{-1}$, the average time between encounters is

$$\Delta t = \left| \frac{\Delta a}{v_0 - v} \right| = \frac{1}{n|v_0 - v|}$$

Thus, the rate is

$$r_{2\mathrm{b}} = n|v_0 - v|$$

and in the general case we just integrate over all possible velocity differences:

$$r_{2\mathrm{b}} = n \int_{-\infty}^{+\infty} |v - v_0| f(v) \mathrm{d}v. \tag{5.1}$$

Note that every 3-body encounter is also a 2-body encounter. A 2-body encounter is also a 3-body encounter iff there exist a third body within ϵ from the pair undergoing the 2-body encounter at the time of their encounter. A single 2-body encounter could count as multiple 3-body encounters, if there are multiple bodies that satisfy the distance condition. Note however that every triple of bodies have a single 3-body encounter, which is the 2-body encounter in which s is the smallest (regardless of ϵ). Therefore, when the average distance between bodies $\Delta a = n^{-1}$ is smaller than ϵ, the environment is "crowded" and the 3-body encounter rate can surpass the 2-body rate.

6. Model for pedestrian motion

We consider a very simple model for pedestrian motion where the velocities are drawn from a double rectangular distribution:

$$f(v)=\begin{cases}0 & v<-V-\frac{1}{2}w\\ \frac{1}{2w} & -V-\frac{1}{2}w<v<-V+\frac{1}{2}w\\ 0 & -V+\frac{1}{2}w<v<\quad V-\frac{1}{2}w\\ \frac{1}{2w} & \quad V-\frac{1}{2}w<v<\quad V+\frac{1}{2}w\\ 0 & \quad V+\frac{1}{2}w<v\end{cases}$$

In this case, the velocity probability density is uniformly distributed in two intervals, one at an environment $\Delta v=w$ around $v=-V$, and the other around $v=V$, and is zero elsewhere. This simple model does not account for standers and joggers (which could be modeled by having additional peaks around $v=0$ and around $v>V$), as well as "binaries" or "multiples" (groups of two or more pedestrians walking together).

Analytical integration of Equation (4.1) is possible in this case, and the result is (v is now the velocity of the reference body):

$$r_{3\mathrm{b}}(v)=4n^2\epsilon\begin{cases}3V+\frac{1}{6}w & 0<v<V-\frac{1}{2}w\\ \frac{1}{w}\left[v^2+(w-2V)v+V^2+2wV+\frac{5}{12}w^2\right] & V-\frac{1}{2}w<v<V+\frac{1}{2}w\\ 2v+V+\frac{1}{6}w & V+\frac{1}{2}w<v\end{cases}$$

The function is of course symmetric around $v=0$, so only positive values are shown. Numerical integration when $f(v)$ is a sum of two normal probability density functions, each centered around $v=\pm V$ shows qualitatively similar results with a constant rate for small v and linear growth of the rate for large v (with smooth transition between these regions). The rate of 2-body encounter is calculated from Equation (5.1), giving

$$r_{2\mathrm{b}}(v)=n\begin{cases}V & 0<v<V-\frac{1}{2}w\\ \frac{1}{2w}\left[v^2+(w-2V)v+(V+\frac{1}{2}w)^2\right] & V-\frac{1}{2}w<v<V+\frac{1}{2}w\\ v & V+\frac{1}{2}w<v\end{cases}$$

The ratio $r_{3\mathrm{b}}/r_{2\mathrm{b}}$ is $\approx 12n\epsilon$ for small velocities and $\approx 8n\epsilon$ for large velocities.

We can also consider the number of encounters per unit length by dividing $r_{3\mathrm{b}}$ by v. The result is a monotonically decreasing functions of v, with an asymptotic values of $8n^2\epsilon$ for $v\gg V$. Thus, the number of encounters per unit length approaches a constant as v increases. A similar behavior is seen for the 2-body encounter, with an asymptotic value of n (which can be expected intuitively).

Figure 6.1 shows the encounter rate as a function of the reference body velocity, where $V=5\,\mathrm{km\,h^{-1}}$, $w=2\,\mathrm{km\,h^{-1}}$, $n=120\,\mathrm{km^{-1}}$ (representing a mildly busy street in Budapest), and $\epsilon=0.5\,\mathrm{m}$ (representing a lower limit on comfortable personal space). For these parameters, the 3-body encounter rate is comparable to the 2-body rate.

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